Instability of Taylor Vortex Flow between Rotating Porous Cylinders

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Abstract

A numerical solution of linear differential equations governing the instability of the Taylor vortex and nonaxisymmetric modes in flow between rotating porous cylinders is presented. Solutions take into account the presence of a radial flow between two rotating cylinders. The critical Reynolds number and corresponding critical axial and azimuthal wavenumber are shown for different values of radius ratio, ratio of angular velocities of the inner and outer cylinders. The results show that not only the critical Reynolds number but the oscillatory onset mode of nonaxisymmetric disturbances can be altered when a radial flow is superimposed on the circular Couette flow. The weak inward flow has a destabilizing effect for wide-gap, co-rotating system of positive and large $\mu$ (ratio of angular velocities $\Omega_1$ and $\Omega_2$), and the weak outward flow has a destabilizing effect for small gap, co-rotating system and all counter-rotating system. The most unstable mode of instability depends not only on the angular speed ratio of both cylinders but also the strength of radial flow.

1. INTRODUCTION

Instability of a viscous flow between two concentric rotating cylinders is of both academic and engineering application interest. Taylor (1923) considered the stability problem both theoretically and experimentally and obtained a perfectly good agreement. He got a criterion for the onset of a secondary motion in the form of cellular toroidal vortices spaced more or less regularly along the axis of the cylinder. Later workers (DiPrima (1955), Chandrasekhar (1961),
Meksyn (1961), Duty and Reid (1964), DiPrima and Swinney (1985) used different approaches to solve this problem for \( \mu \) (ratio of the angular velocities of the two cylinders) very negative and large. They all solved this problem for axisymmetric disturbances with small-gap assumption, where mean flow can be replaced by its average value. Krueger et al. (1966) went to consider the fully linear Taylor problem for negative \( \mu \), and found that in narrow-gap approximation, when \( \mu \) is less than -0.78, the most unstable disturbance is no longer axisymmetric but nonaxisymmetric. As \( \mu \) decreases below this value, the most unstable mode changes from \( m \) (azimuthal wavenumber) = 0 to 1 but \( \mu \) is the ratio of the radial velocities of the two cylinders. For example, Coles (1965) and Snyder (1968). Nonaxisymmetric disturbances usually corresponding to unsteady onset shows a very different feature from steady onset, which is well assumed in instability problems of the closed type, i.e. Taylor problem and Benard problem. Perturbation growth rates \( (\sigma) \) are usually assumed to be zero in the formulation of the two problems, but \( \sigma \) is actually not zero for unsteady onset. This makes the problem more complicated and difficult to be solved.

Another interesting problem in this case is the effects of an additional flow on the stability of a viscous flow between two rotating cylinders. For example, Chandrasekhar (1960), DiPrima (1960), Chung and Astill (1977), and Babcock et al. (1991) showed that an axial flow in the annulus stabilizes the circular Couette flow. At the same time, the stability problem for viscous flow between porous cylinders with a radial flow also has been widely investigated. Chang and Sartory (1967,1969), Min and Lueptow (1994), Kolyshkin and Vaillancourt (1997) predicted that inward radial velocity and strong outward velocity stabilize the flow, while weak outward velocity destabilizes the flow. Bahl (1970) made the narrow-gap approximation and concluded that an radially inward velocity stabilizes the flow, while an radially outward velocity destabilizes the flow. Similar type of flow occurs during the dynamic filtration process in biotechnology. By using a rotating inner porous cylinder and stationary nonporous outer cylinder, as a suspension of fluid and particles moves axially between the cylinders, the filtrate passes radially through the wall of the inner cylinder and the concentrate is collected at the exit end of the annulus, opposite to the entrance end (Kroner et al. 1987, Wronski et al. 1989). The porous cylinders in this paper can be treated as a rotating shear filter in which the suspension is filled from outer porous wall and filtrate is collected inside the inner porous wall.

2. FORMULATION

Let \((r, \theta, z)\) denotes the usual cylindrical coordinates, and let \((u_r, u_\theta, u_z)\) be components of velocity in the positive \((r, \theta, z)\) directions, respectively. We consider an incompressible viscous fluid in absence of a body force which contained within two infinite long cylinders, both cylinders are made of porous material, with the \( z \)-axis as their common axis. If we let \( r_1, r_2, \Omega_1, \Omega_2 \) denote the radii and angular velocities of the inner and outer cylinders, respectively. The flow is driven not only by the rotation but also the radial velocity between the two cylinders. The Navier-Stokes equations and continuity equation admit a steady solution in terms of the velocities:

\[
(u_r, u_\theta, u_z) = (\alpha v/r, Ar^{a+1} + B/r, 0) \tag{1}
\]

here \( \alpha = u_r r_1 / \nu \) is the radial Reynolds number, where \( u_r \) denotes the radial velocity through the inner porous cylinder, with a positive value stands for a outward velocity from the center of cylinder. The constants are

\[
A = -\frac{\Omega_r \eta^2 (1 - \mu / \eta^2)}{r_2^2 (1 - \eta^{a+2})}, B = \frac{\Omega_r r_2^2 (1 - \eta^{a+2})}{1 - \eta^{a+2}} \tag{2}
\]

where \( V \) is the ratio of angular velocities and \( \mu \) is the ratio of the radii, and \( \alpha = -2 \) is a avoiding point.

To study the stability, following the deviation of Krueger et al. (1966), of this flow we superpose a general disturbance on the basic solution, substitute in the equations of motion, continuity equation and neglect the high order terms. Since the coefficients in the resultant disturbance equations depend only on \( r \), it is
possible to seek solutions for velocities, pressure perturbations

\[ u'_\alpha = d_1 \Omega_1 u (r) e^{i \alpha g(r) + \lambda z} \]
\[ u'_\sigma = d_1 \Omega_1 v (r) e^{i \alpha g(r) + \lambda z} \]
\[ u'_\tau = d_1 \Omega_1 \tau (r) e^{i \alpha g(r) + \lambda z} \]
\[ p' = \rho v \Omega_1 p (r) e^{i \alpha g(r) + \lambda z} \]

where \( x = (r - r_\sigma)/(r_\tau - r_\sigma) \).

It is noted that \( m \) must be an integer for the reason that the azimuthal wave must be countable and the \( \lambda \) must be real for the solution must be bounded at \( z \to \pm \infty \), while the parameter \( \omega \) is in general complex.

If we replace the pressure term \( \pi(x) \) by introducing a new variable \( X(x) \) defined by

\[ \pi(x) = D^*u(x) - X(x) \]

where

\[ \pi(x) = \frac{d}{d_1} p(x) \]
\[ r = r_\sigma + x d_1, D = d/d_1, D^* = d/dx + \xi(x) \]
\[ \xi(x) = \delta / (1 + \delta x), \delta = d_1 / r_\sigma, d_1 = r_\tau - r_\sigma \]

In order to get the sets of ODE's, it is convenient to let \( Y = D^*v(x), Z = Dw(x) \), and make use of continuity equation to replace \( D^*u \), we obtain the following system of six first-order equations

\[ Du = -im \xi(x) v - i a w - \xi(x) u \]
\[ Dv = Y - \xi(x) v \]
\[ Dw = Z \]

\[ DX = M(x) u + 2im \xi^2(x) v - Ta\Omega^* v \]
\[ - 2a \xi^2(x) u - i a \alpha \xi^2(x) v - i a \alpha \xi(x) w \]

\[ DY = [M(x) + m^2 \xi^2(x)] v - im \xi^2(x) X + ma \xi(x) w \]
\[ \alpha \xi(x) Y - 2im \xi^2(x) \alpha + (\alpha + 2)Ta A/\xi^2(x) u \]
\[ DZ = [M(x) + a^2] w - i a X + am \xi(x) v \]

where

\[ M(x) = a^2 + m^2 \xi^2(x) + i[\sigma + mTa \Omega^*] \]
\[ \Omega^* = A/\xi^2(x) + B^2 \xi^2(x) \]
\[ A^* = \frac{d^*}{\Omega_1 r_\sigma}, B = \frac{1}{\Omega_1 r_\sigma d_1} B^* \]

\[ a = \lambda d_1, \sigma = \omega d_1 / \nu, Ta = \Omega_1 r_\sigma d_1 / \nu \]

The parameters presented above \( Ta, a, \alpha \) and \( \sigma \) are rotational Reynolds number, axial wavenumber, radial Reynolds number, and perturbation growth rate, respectively.

The boundary conditions are

\[ u = v = w = 0 \text{ at } x = 0,1 \]

The eigenvalue problem formed by (6)-(11) together with boundary condition (13) can be of the form

\[ F(\mu, \eta, m, a, \sigma, \alpha, Ta) = 0 \]

The marginal state is characterized by \( \sigma_i \), the imaginary part of \( \sigma \), equal to zero. In this nonaxisymmetric case, the exchange of stability, i.e. \( \sigma \) is assumed to be identically zero, can not be assured to be \( a \text{ priori} \), since the oscillatory mode is its nature for nonzero azimuthal waves. Hence not only \( \sigma \) is complex, but all the variables \( u, v, w \) are also complex. This makes the problem (6)-(11) change to a set of 12 first-order ODE's with the same boundary condition (13) to another form

\[ G(\mu, \eta, m, a, \sigma, \alpha, Ta) = 0 \]

where \( G \) is real-valued functionals with \( \sigma_r \) to be determined. For given value of \( \mu \) and \( \alpha \), we seek for the minimum real positive \( Ta \) over real \( a \) and integer \( m \geq 0 \), for which there exists a solution for (15) with \( \sigma_r \) not equal to zero.

The value of \( Ta \) to be solved is critical Reynolds number \( Ta_c \), for assigned parameters stated above, the disturbances then are determined by the corresponding values \( \alpha_c \) and \( m \), which are called critical axial and azimuthal.
wave numbers, respectively.

The eigenvalue problem (15) can be solved as Walowit et al. (1964) via a Galerkin method. Here we solve the two-point eigenvalue problem by shooting technique, which makes the problem an initial-value problem, together with a unit-disturbance method. This method has been widely used by, for example, DiPrima (1960), Harris and Reid (1964), Krueger et al. (1966), Soundalgekar et al. (1990), Min and Lueptow (1994), and Kong and Liu (1994) for similar hydrodynamic stability problems. Procedures of Krueger et al and Min et al are preferred here so as to see the details. A faster convergence of the iteration, which determines the critical Reynolds number, $\sigma_r$ and corresponding axial wave number $a_c$, is obtained by utilizing a hybrid algorithm developed by Powell on the basis of a modified Newton-Raphson scheme, as well as the steepest-descent iteration instead of bivariate used by Krueger et al.

3. DISCUSSION

In Fig. 1, which confirms the results of Krueger et al. (1966): "......for $R_1/R_2 = 0.95$, $\Omega_1/\Omega_2 = -1.0$, the wave number in the azimuthal direction of the critical disturbance is $m = 4...",$ at the same time, it is clear that strong outward velocity is benefit to the axisymmetric disturbances, and the nonaxisymmetric disturbances of higher modes prefer strong inward velocity. In fact, our results show that at $\eta = 0.95$, if $\mu \leq -0.8$, the most unstable disturbances are indeed nonaxisymmetric ones, and the modes depend upon the $\mu$, $\eta$ and $\alpha$. Similar phenomenon can be found in the research of radial temperature gradient acts on the region between two coaxial cylinders by Kong and Liu (1994).

Since researches by Min and Lueptow (1994), Kolyshkin and Vaillancourt (1997) are all focused on the instability arising from counter-rotating mechanism, we will extend their results by adding the co-rotating effects here.

In Fig. 2, we can find for small gap cases ($e.g. \eta = 0.95, 0.85$) with positive and large $\mu$ (we take $\eta = 0.95, \mu = -1.0$ here), the incipient instabilities still appear in $\alpha > 0$, but for wide gap ones, lowest Taylor numbers appear in $\alpha < 0$, we have verified that all modes appear in this range are all axisymmetric. In Fig. 3, for $\mu = -1.0$, we change the gap sizes from $\eta = 0.75 \sim 0.95$, the radial inward flows can not destabilize the flow field any more, then we can conclude that the weak inward flow has a destabilizing effect for wide-gap, co-rotating system of positive and large $\mu$, and the weak outward flow has a destabilizing effect for small gap, co-rotating system and all counter-rotating system.

![Fig. 2 The variation of critical Reynolds No. ($T\alpha$) for various $\eta$ with $\mu = 0.4$](image)

![Fig. 3 Relationship between critical Reynolds No. ($T\alpha$) and radial Reynolds No. ($\alpha$) for $\mu = -1.0$.](image)

As earlier concept: in the absence of the radial velocity, the critical Reynolds number of
the flow field is inversely proportional to gap width. But since the radial velocity is present, the smallest Reynolds numbers may not be wide gap ones. In Fig. 3, we can find that the inward flow enhances the critical Reynolds numbers for all gap sizes, but as the \( \eta \) decreases, the effect increases rapidly. Therefore, on the influence of inward flow, the wide gap modes may show higher critical Reynolds numbers than small gap ones. From the results obtained by Min and Lueptow, Kolyshkin and Vaillancourt, the strong outward flow has the stabilized effect, too. And the same phenomenon may occur in large, positive \( \alpha \).

If we check Fig. 3 for \( \alpha = 0 \), the onset modes for \( \eta = 0.95, 0.90, 0.85, 0.80, 0.75 \) are 4,3,3,2,2, respectively. This also coincides with the result obtained by Krueger et al. in their Table 2. For most of cases, at the same radial flow strength, the unstable modes for larger \( \eta \) s are always larger or equal to smaller \( \eta \) s, and when the inward flow becomes stronger, the unstable modes become higher and higher, but the unstable modes will decrease when the outward flow becomes stronger. Unlike critical Reynolds numbers, the critical axial wave numbers show less regularity in the absence of radial flow, but since the effect of radial inward flow becomes stronger, the critical axial wave number increases as its gap size increases, and the axial wave number decreases as the \( \alpha \) increases, but instead of decreasing, the axial wave number will increase while the radial outward flow becoming more stronger (see Fig. 4).

4. CONCLUSIONS

Instability of Taylor vortex flow between rotating porous cylinders is indeed affected by a radial velocity. The nonaxisymmetric mode can be altered by the strength of the radial velocity and the angular speed ratio of both cylinders. For strong inward flow, nonaxisymmetric modes prevail, but for strong outward flow, axisymmetric modes prevail.

The weak inward flow has a destabilizing effect for wide gap, co-rotating system of large \( \mu \), and the weak outward flow has a destabilizing effect for co-rotating system of small \( \mu \), and all counter-rotating system. Both strong inward flow and strong outward flow have stabilizing effect. Whether the gap is wide or not, for large \(-\mu\), the nonaxisymmetric modes predominate. And in the presence of inward flow, wide gap systems may show stronger stability than small gap ones.

We wish that the results obtained from present study could be a little benefit to the research of dynamic filtration process in biotechnology.

REFERENCES


between rotating coaxial cylinders with fully developed axial flow," Journal of Fluid Mechanics, Vol. 81, pp. 641