Using Eulerian-Lagrangian Method of Fundamental Solutions to Solve the Shallow Water Equations

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ABSTRACT

The objective of this paper is to develop a novel meshless numerical model to solve the system of shallow water equations. The proposed Eulerian-Lagrangian method of fundamental solutions based on the diffusion fundamental solution and the Eulerian-Lagrangian method are able to easily handle the system of hyperbolic equations. The coupled shallow water system will be first transferred to the system of the pure advection equations, and then the solutions are approximated by the proposed meshless numerical scheme. Furthermore the proposed numerical method free from mesh and numerical quadrature can transform the physical variables between the Eulerian and Lagrangian coordinates. There is a numerical tests for validating the proposed numerical scheme and the numerical results are compared well with the analytical solutions. The problem governed by the shallow water equations was analyzed by the proposed meshless method and the results are compared with the other solutions. Therefore the proposed Eulerian-Lagrangian method of fundamental solutions is a promising numerical solver for the system of shallow water equations.

Key words: Shallow water equations; Eulerian-Lagrangian method of fundamental solutions; meshless numerical method; pure advection equation

使用尤拉-拉格朗日基本解法
求解淺水波方程式

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摘要

本文的目的在於發展新穎的無網格數值方法用以求解淡水波方程式之系統問題。本文所提出
的尤拉-拉格朗日基本解法可以輕易的駕馭雙曲線型系統方程問題。首先，耦合的淺水方程式系
統會被轉換成非耦合之對流方程式系統，之後近似解由本文所提出之數值方法解出。本文所提
出之數值方法不需要網格或數值積分，而且可以輕易的將物理變數在尤拉與拉格朗日座標系間
轉換。本文列出一個數值實驗用以測試所提出的數值方法，其數值解與解析解比較結果良好。
之後，淺水波問題被選來證明本文所提出之無網格數值方法概念。其數值解亦與解析解比較結
果良好。藉此證明本文所提出之尤拉-拉格朗日基本解法對於求解淡水系統方程式而言是一個有
潛力的數值方法。

關鍵字：淡水方程式、尤拉-拉格朗日基本解法、無網格數值方法、純對流方程式
1. Introduction

With the improvement of computer system, the computation fluid dynamics has become a popular and stable technique for fundamental researches and engineering applications. Although many numerical methods are developed, solving the system of the hyperbolic equations is still an essential topic, such as stress wave in elastics, water wave in the ocean engineering and gas dynamics simulation, etc. These problems are generally governed by the hyperbolic partial differential equations. The classical numerical schemes, such as total variation diminishing (TVD) scheme (Harten, 1983), essential non-oscillation (ENO) scheme (Harten et al., 1987), weighted essential non-oscillation (WENO) scheme (Jiang and Shu, 1996) and characteristics-based-split (CBS) scheme (Zienkiewicz et al., 1999), are developed to solve the first-order hyperbolic-type partial differential equation. We proposed in this paper a novel numerical scheme which is a meshless solver for hyperbolic partial differential equations. The numerical method does not need mesh generation and numerical integral. The so-called meshless method can be classified into the domain-type and boundary-type numerical solvers for the partial differential equations. The present scheme, method of fundamental solutions (MFS), belongs to the boundary-type meshless method. In previous researches (Kupradze and Aleksidze, 1964), the concept of MFS was proposed. Then the elliptic boundary value problems were solved by MFS (Mathon and Johnston, 1977). In time-dependent problem, the diffusion fundamental solution was used to solve the homogeneous diffusion problems (Young et al., 2004).

The MFS is able to solve the time-dependent problem with the diffusion fundamental solution, but the advection phenomenon cannot be simulated directly by MFS. The Eulerian-Lagrangian method (ELM) is a useful tool to describe the advection phenomenon. The boundary element method (BEM) with ELM was proposed to solve the advection-diffusion problem (Young et al., 2000). However, the mesh generation and numerical integration are needed in the implementation of BEM. The Eulerian-Lagrangian method of the fundamental solutions (ELMFS) was presented for solving the multi-dimensional advection-diffusion equation (Young et al., 2006). The ELMFS was extended to approximate the pure advection equation (Gou and Young, 2005) and they proved that the ELMFS is a powerful and meshless numerical method for pure advection problems. The advection-diffusion equation was used to approximate the hyperbolic equation with weak diffusion effect.

In this paper a novel hyperbolic system solver based on the ELMFS is developed, and further explains the details of the ELMFS and the decoupled procedure. In section of numerical tests, the present scheme was adopted to solve the problem of linear hyperbolic system. Finally the shallow water problem was selected to show the application of the proposed scheme. All of the numerical results are compared well with the analytical solutions. Some conclusions and discussion based on the numerical results are drawn in the last section.

2. Governing equation

The hyperbolic system was considered with no source term in the one-dimensional domain. The system of the hyperbolic partial differential equation can be written as:

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0
\]  

(1)

Where respectively \(t\) and \(x\) are time and space coordinates. In Eq. (1), \(U\) and \(F(U)\) are the vectors of the conserved variables. The Eq. (1) can be written as the quasi-linear form:

\[
\frac{\partial U}{\partial t} + J(U)\frac{\partial U}{\partial x} = 0
\]  

(2a)

Where \(J(U)\) is the Jacobian matrix corresponding to the flux \(F(U)\). The definition of the Jacobian matrix can be expressed as:
\[ J(U) = \frac{\partial F_j}{\partial U_j} \]

(2b)

From the Jacobian matrix, \( J(U) \), the eigenvalues of the matrix can be obtained by the following definition:

\[ \lambda \left[ J - \lambda I \right] = 0 \]

(3)

Where, \( \lambda \) is the eigenvalue of the Jacobian matrix, \( I \) is the identity matrix. The eigenstructure can be written as:

\[ J = R\Lambda R^{-1} \quad \text{or} \quad \Lambda = R^TJR \]

(4a)

Where, \( \Lambda \) is the diagonal matrix and \( R \) is a matrix which bases on the right eigenvectors. Assuming the inverse matrix \( R^{-1} \) exists, we transferred the system variable as the following relationship:

\[ Q = R^T U \quad \text{or} \quad U = QR \]

(4b)

From the eigenstructure relationships, the coupled system Eq. (1), can be transferred to the decoupled system as the following:

\[ \partial_t \lambda + \Lambda = \partial_t \Omega \]

(5)

As in the following formulation, the present method will introduce an artificial viscous term into Eq. (5):

\[ \frac{\partial Q}{\partial t} + \Lambda = \frac{\partial Q}{\partial t} + \kappa \nabla^2 Q \]

(6)

The diffusion operator in Eq. (6) will be described by the diffusion fundamental solution.

3. Numerical method

From the diffusion equation, the fundamental solution of the linear diffusion equation is governed by the following equation:

\[ \frac{\partial G}{\partial t} + \kappa \nabla^2 G + \delta(x-x')\delta(t-t') \]

(7)

Where, \( G \) is free-space Green’s function or the fundamental solution, \( x \) and \( x' \) are the space location of field and source points, \( t \) and \( t' \) are the time coordinates of the field and source points respectively, \( \delta(\cdot) \) is the Dirac delta function and \( \kappa \) is the diffusion coefficient. The distributions of field and source points are shown in Fig. 1(a). By taking the Fourier and Laplace transforms, the free-space Green’s function of the diffusion equation can be obtained as:

\[ G(x,t;x',\tau) = \frac{1}{(2\pi \sigma_0)^d} \frac{\kappa}{4\pi \sigma(t-t')} \exp \left( -\frac{(x-x')^2}{4\sigma^2(t-t')} \right) \]

(8)

Where, \( N_{\text{dim}} \) is dimensional of space and \( H(\cdot) \) is the Heaviside step function. Since the diffusion fundamental solution satisfies the homogeneous diffusion equation, the solution of diffusion equation can be assumed as a linear combination of the fundamental solution for diffusion operator. According to the MFS, the numerical solutions of the diffusion equation will be written as the following form:

\[ Q(x,t) = \sum_{j=1}^{M} \alpha_j G(x,t;x_j,\tau_j) \]

(9)

Where, \( M \) is the number of source points, \( \alpha_j \) are the undetermined coefficients. A linear matrix system can be formed by collocating the initial and boundary conditions.

\[ \begin{pmatrix} \frac{\partial Q}{\partial t} + \lambda Q \end{pmatrix} \| \begin{pmatrix} 0 \\ \kappa \nabla^2 Q \end{pmatrix} \| = \begin{pmatrix} B \end{pmatrix} \]

(10)

The vector \( B \) is obtained from the initial and boundary conditions. After solving the linear matrix system, the coefficients \( \alpha \) can be obtained. Then the solution of diffusion equation can be expressed by Eq. (9). From the Fig. 1 (a) and (b), the point distribution and the characteristic paths are shown in the time-space domain. In the ELM, the advection velocity can be expressed in terms of the spatial and time increments as follows:

\[ \frac{dx}{dt} = \nu \Rightarrow x^t = x^{t+\nu} - \nu \Delta t \]

(11)

The Fig. 1 (b) is the line of characteristic path which transport the scalar quintiles. If the functional value at point \( A_1 \) is required, the position of \( A_2 \) can be traced as Eq. (11). According to the material derivative and the linear diffusion equation, the solution along the characteristic path must satisfy the diffusion operator as Eq. (6). After the diffusion process is calculated by the MFS, the functional value at \( A_3 \) can be found. Points \( A_2 \) and \( A_3 \) points are located at the same position and different time levels in the time-space domain. If the functional value at point \( A_1 \) is replaced by the diffusion result at point \( A_3 \), the advection-diffusion results at \( t = (n+1) \Delta t \) level can be obtained. Combining the ELM and MFS, we can solve...
the advection-diffusion equation.

4. Numerical results

4.1 The linear hyperbolic system

In this numerical test, the linear hyperbolic system problem was selected to prove the feasibility of the proposed scheme. In this problem, the hyperbolic system can be written as follows:

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in} \quad 0 \leq x \leq 4\pi \]  

(12)

The initial condition:

\[ u(x,t)|_{t=0} = \sin(x) \quad \text{and} \quad u_x(x,t)|_{t=0} = \cos(x) \]  

(13)

The boundary condition:

\[ u(x,t)|_{x=0} = \frac{1}{2} \sin(4\pi - 4t) + 2\cos(4\pi - 4t) \]  

\[ -\frac{1}{2} \cos(4\pi x) - \sin(4\pi x) \]  

(14)

The analytical solution:

\[ u_1(x,t) = \frac{1}{2} \sin(x - 4t) + 2\cos(x - 4t) \]  

\[ -\frac{1}{2} \cos(x + 4t) - \sin(x + 4t) \]  

(15)

In this case, the computer calculation used 41 pints, time-interval \( \Delta t = 0.002 \) and the diffusion coefficient set as an extremely small constant \( K = 10^{-6} \). The Fig. 2 (a) and (b) described the time evolution at \( x = 4 \) and 8, respectively. The Fig. 3 show the root-mean-square (RMS) error (ERM) time evolution, the RMS error was defined as:

\[ E_{\text{RMS}} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( \theta_{\text{analytical}} - \theta_{\text{numerical}} \right)^2} \]  

(16)

In this case, the setup effect is clear shown in Fig. 3 from \( t = 0 \) to \( t = 2 \). After \( t = 2 \), the RMS errors become stable oscillation. In this numerical result, the RMS errors almost surround with \( 10^{-2} \). Hence, it is convinced the ELMFS is a highly stable scheme.

4.2 The shallow water problem

In the final test, the linear shallow water problem was selected to display the application of the proposed scheme. A rectangular channel with length \( L = 160 \) and static depth \( H = 2 \) are considered. The amplitude of wave is set as \( \gamma_0 = 0.1 \) and the period is \( T = 200 \). The linear shallow water equations can be written as follows:

\[ \frac{\partial \eta}{\partial t} +Hu = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = 0 \quad \text{in} \quad 0 \leq x \leq L \]  

(17)

Where, \( \gamma \) is the amplitude of the free surface, \( u \) is the velocity, \( g \) is the gravitational acceleration. The initial condition in this problem can be written as:

\[ \eta(x,t)|_{t=0} = \frac{\eta_0 \cos(\beta x)}{\cos(\beta L)} \quad \text{and} \quad u(x,t)|_{t=0} = 0 \]  

(18)

The boundary condition:

\[ u(x,t)|_{x=0} = 0 , \eta(x,t)|_{x=L} = \eta_0 \cos(\alpha x) \]  

\[ u(x,t)|_{x=L} = \frac{\eta_0}{HL} \tan(\beta L) \sin(\alpha x) \]  

(19)

Where, \( \omega = 2\pi / T \) and \( \beta = \omega^2 / g H \). The analytical solution (Lynch and Gray, 1978) can be written as:

\[ u(x,t) = \Re \left( \eta_0 \frac{\cos(\beta x)}{\beta L} e^{i\omega t} \right) \]  

\[ \eta(x,t) = \Re \left( \frac{\eta_0}{\beta H} \sin(\beta x) e^{i\omega t} \right) \]  

(20)

Where, \( \Re(\cdot) \) is the real part of the complex number. In this case, the computer calculation used 21 pints, time-interval \( \Delta t = 1 \) and the diffusion coefficient set as an extremely small constant \( K = 10^{-6} \). The Fig. 4 (a) and (b) described the time evolution of amplitude at \( x = 0 \) and 80, respectively. The Fig. 5 show evolutions of the maximum absolute error of amplitude and velocity. In this numerical test, the RMS errors are almost small than \( 10^{-2} \). Therefore, it is convinced the ELMFS is a highly accurate and stable scheme.

5. Conclusion

In this paper, a novel numerical method based on the ELMFS was developed to approximate the solution of the hyperbolic partial differential equations. In ELM, it is easy to transform the solutions between the Eulerian and the Lagrangian coordinates alone
characteristic path, and the model performs well for both the advection- and diffusion-dominated problems. The numerical integration is not needed for MFS calculation on the space domain. The proposed ELMFS scheme based on the diffusion fundamental solution and the method of characteristics can be easily used for handling the single hyperbolic equation or system problem. The present scheme simultaneously becomes a useful tool for controlling the artificial diffusion effects by diffusion coefficient. First, the system of hyperbolic equations will be decoupled for the system of pure advection equations, and then the solutions are approximated by the proposed meshless method. Then, the next time-step unknown variables are easily obtained by the proposed scheme along the characteristics. The numerical tests about the linear hyperbolic system are selected for validating the proposed numerical scheme. The numerical results also compared well with the analytical solutions. Therefore, it is evident that the proposed ELMFS is a promising numerical tool for solving system of hyperbolic equations.

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References

Fig. 1 The point distribution and the characteristic path. (a) The field and source points (b) The characteristic path.

Fig. 2 The solution of the linear hyperbolic system at (a) $x = 4$ and (b) $x = 8$.

Fig. 3 The time evolution of RMS error.

Fig. 4 The time evolution of the amplitude (a) $x = 0$ (b) $x = 80$.

Fig. 5 The time evolution of maximum absolute error.