A second-order analytic approximation of McCowan’s solitary waves

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ABSTRACT

A second-order approximation for McCowan’s solitary waves is proposed in the paper using collocation method. Better agreement of wave profiles and wave speeds of different amplitudes with highly accurate analytical solution than the first-order approximation indicates the accuracy of the proposed approximation. An improper integration of the second-order approximation is used to calculate wave drift. The method is examined valid to estimate particle drifts by comparing with Fenton’s ninth-order approximation. The present computation shows that the particle at the surface flows by a distance about three times of water depth and marches faster by one and half times than that at the bottom for large solitary waves.

Keywords: collocation method; McCowan’s solitary wave; particle drift; second-order approximation

1. Introduction

The experimental studies on the steady finite amplitude solitary wave were first reported by Russell (1844) who described solitary waves moving with almost constant form. For decades an overall approximation to a two-dimensional, irrotational solitary wave which is of truly constant form and celerity has not been obtained until Boussinesq (1871) and Rayleigh (1876). Some researchers, such as McCowan (1891) and Korteweg and de Vries (1895), Laiton (1960), Grimshaw (1971), Long (1956) etc. proposed low-order approximations for solitary waves. Fenton (1972) gave explicit expressions for some physical quantities, including the surface profile, speed of the wave and the drift of fluid particles.

All analytical approximations described above are applicable for finite-amplitude solitary waves. Some direct numerical calculations with high accuracy for large waves even for the maximum wave, in form of exact integral equation, or integro-differential equation, or a series expansion were proposed. A good review of these investigations can be found in Miles (1980), Okamoto and Shoji (2001) and Wazwaz (2009).
Although McCowan’s solitary theory is the first-order approximation, McCowan (1891) provided the velocity potential and stream function to describe the velocity of any particle in fluid and overall investigations on the dynamic and kinematic properties of solitary waves, including particle trajectory and drift. The computation is required to have particle drifts of solitary waves. Alternative method of integrating on the horizontal velocity of a particle in steady state proposed by Fenton (1972) is used to calculate the drifts of particles at the surface, mean depth and on the bottom. The obtained wave drifts are compared with those of Fenton’s ninth-order approximation for examining the accuracy of the proposed approximations. The present paper follows the previous work (2013) and gave a second-order approximation. The application of the second-order approximation was examined valid. Wave drift was discussed in the fourth section.

2. Methodology

2.1 Formulation

A two-dimensional solitary wave of permanent form propagating at a constant speed, c, on the surface of water over a flat bed is considered in the paper. The Cartesian coordinates (X, Y) are used to describe the wave motion. The Y-axis is vertical, and the X-axis represents the direction of wave propagation. The origin is set on the flat bed. A uniform current with a velocity equal and opposite to that of the wave propagation is assumed to impose on the flow field so that the wave motion is then in a steady state. An illustration of surface elevation of a solitary wave and reference coordinates is shown in Fig. 1.

When the fluid is assumed to be incompressible and the wave motion is irrotational, the flow field can be described by a velocity potential or a stream function. Boussinesq (1871) and Rayleigh (1876) independently gave an expression for this motion that can be modified to be in term of $x+iy$:

$$F(x+iy) = \sum_{n=1}^{\infty} \left( \tan \frac{m(x+iy)}{2} \right)^{2n-1}, \quad m(x+iy) < \pi$$  \hspace{1cm} (1)

where $a_{2n-1}$ are the coefficients which can be determined by the method of successive approximation and $m$ is so called as straining parameter which is similar to the wave number of periodic Stokes waves.

When the two terms in Eq. (1) was retained and higher order terms were neglected McCowan (1891) obtained the leading stream function

$$\phi = ca_1 \frac{\sinh mx}{\cos my + \cosh mx}$$  \hspace{1cm} (2)

and

$$\varphi = -ca_1 \frac{\sin my}{\cos my + \cosh mx}$$  \hspace{1cm} (3)

The second term of Eq. (1) gives

$$\phi = ca_2 \frac{\sinh^2 mx - 3 \sin^2 my \sinh mx}{(\cos my + \cosh mx)^2}$$  \hspace{1cm} (4)

and

$$\varphi = ca_3 \frac{3 \sin my \sinh^2 mx - 3 \sin^3 my}{(\cos my + \cosh mx)^3}$$  \hspace{1cm} (5)

The stream functions at both bottom and free surface are constant. Setting $y=0$ on Eq. (1) yields $\varphi = 0$ at the bottom. When Eq. (1) is specified by $x \rightarrow \pm \infty$, the value of $\varphi$ is verified to be $ch$ on the free surface at limiting ends. When $y = h + \eta$ and $\varphi = \epsilon h$ are taken on Eq. (1) the free surface above the still depth at any position $x$ will be expressed by

$$\eta(x) = a_2 \frac{\sin h + \eta}{\cos h + \cosh mx} + a_3 \frac{3 \sin h + \eta \sinh^2 mx - 3 \sin^3 h + \eta}{\cos h + \cosh mx}$$  \hspace{1cm} (6)

The dynamic boundary condition at the surface is
the Bernoulli’s equation indicating the conservation between the kinetic energy and potential energy and follows

\[ g\eta + \frac{1}{2}(u^2 + v^2) = \frac{c^2}{2} \]

(7)

where the velocity of a particle is obtained by the definition of \( (u,v) = (-\psi_y, \psi_x) \) where the subscripts indicate partial derivatives.

When \( a \) and \( h \) are specified for a solitary wave, there are four parameters, that are the wave celerity and the straining parameter and two subsidiary constants \( (\alpha_i, \alpha_3) \) to be determined.

### 2.2 Collocation method

Expanding the leading term of Eq. (2) and Eq. (3) in powers of \( \eta \) and neglecting all terms beyond \( \eta^2 \) McCowan (1891) obtained the first-order approximation for these three parameters.

Chang and Liou (2013) developed a renormalized approximation and collocation method for the parameters. Comparisons of these approximations with highly accurate analytical solutions for wave profiles, wave speed and horizontal drifts of particles show that the approximation using collocation method is the best among three first-order approximations.

Collocation method is to choose some points to fit the equations for determining the unknown parameters. The number of fitted equations is the same of that of unknowns. If we choose two collocations points at the surface to satisfy Eq. (6) and Eq. (7) four fitted equations are obtained. However, the chosen points at the surface except the crest are still unknown and become extra unknowns. Another more collocation point should be added to fit the boundary conditions. Using Collocation method for the McCowan’s second-order approximation three collocation points including the crest are required to have six fitted equations that form a closed system for solving six unknowns, that are \( m, c, \alpha_1, \alpha_3 \) and two surface points. The roots of this closed system can be numerically solved by the Newton’s method with suitable initial guesses.

The first-order approximation by Boussinesq (1876) explicitly shows \( m = \sqrt{3a}/h \) and then \( c \) and \( \alpha_1 \) can be directly computed by Eq. (6) and (7). The value of \( \alpha_3 \) can be set zero or 0.1 \( \alpha_1 \). Two extra collocation points are chosen as \( \eta \approx 8.0 \) and 0.3 \( \alpha_1 \) of which the locations are estimated by renormalization formula of the first-order approximation. Some other choices are of course possible. It is not the goal here to investigate the effect of varying the collocation points. The proposed second-order approximation is here called as MC2 differing from the first-order approximation, MC1 of Chang and Liou (2013).

### 3. Mathematical validity

The proposed approximation exactly satisfies the governing equation and bottom boundary condition but approximately satisfies the dynamic boundary condition and the kinematic boundary condition except those collocation points. In this paper the validity of these approximations is pursued by comparing with highly accurate solutions for wave profiles and wave speeds and by examining the satisfaction of the boundary conditions at surface at any position for the approximation. Two more accurate solutions are available than these approximations. Those are Fenton’s ninth-order approximation (1972) and Wu’s 18th-order approximation (2005b) by the unified perturbation theory, which is hereafter denoted by UPT in this paper.

#### 3.1 Wave profile

Wave profiles at any position can be numerically computed by Eq. (6) or by Eq. (7), which is a nonlinear equation with three parameters. Fenton’s ninth-order solution was compared with Byatt-Smith’s numerical solution, that is commonly accepted for an “exact” solution, to be perfectly adequate for \( a/h \) up to 0.5, but larger underestimation for higher waves. A case of \( a/h=0.752 \) was the highest amplitude in comparisons of wave profiles with those of Byatt-Smith. Here we choose the case of \( a/h=0.75 \) for
comparing wave profiles obtained from solving Eq. (6) for MC1 and MC2 with that of UPT. The result is shown in Fig. 2. It is seen from Fig. 2 that Wave profiles of MC1 and MC2 are approximate and slighter lower than that of UPT for \( x/h < 1.5 \), but insignificantly higher than that of UPT for \( x/h > 1.5 \).

![Fig. 2 Comparison of wave profiles with that of Wu et al. for a/h=0.75.](image)

If Eq. (6) is selected to compute wave profiles at all positions the dynamic boundary condition at the surface can not be exactly satisfied for these wave profiles so that Eq. (7) has a residual, denoted by \( \text{Err} \), of Bernoulli’s equation. Figure 3 shows the relative residual of MC1 and MC2 approximations, which is defined by the ratio of residual of Bernoulli’s equation to the wave amplitude within \( x/h < 5 \), for \( a/h=0.75 \). Figure 3 shows that MC2 has smaller relative residuals within whole ranges, particularly at the tails of the solitary wave, than MC1.

![Fig. 3 Relative residual at any position for a/h=0.75.](image)

In order to examine the satisfaction of Bernoulli’s equation for possible wave amplitudes, maximum relative residual of the approximations for a solitary wave is selected and plotted in Fig. 4 for small-amplitude wave up to the maximum wave.

![Fig. 4 Maximum relative residuals of Bernoulli’s equation against different wave amplitudes.](image)

The maximum solitary wave is at a condition of about \( a/h \approx 0.83 \) (seeing Wu et al. (2005a)). MC2 significantly having smaller maximum relative residuals for all wave amplitudes than MC1 shown in Fig. 4 indicates MC2 has higher accuracy than MC1.

### 3.2 Wave speed

Wave speed is a key parameter to show dynamic properties of solitary waves. Therefore, wave speed is generally taken as an examination on the accuracy of a wave approximation. Figure 5 is the comparison of relative wave speeds of MC1 and MC2 with that of UPT for \( a/h \leq 0.83 \). The relative wave speed is defined by the ratio of calculated wave speed to the wave speed of the linear long wave theory, \( c_0 = \sqrt{gh} \). Top plot of Fig. 5 shows the obtained wave speeds and bottom plot indicates deviations of wave speed of two approximations from that of UPT. From Fig. 5 it is seen that MC2 is much fitter to that of UPT than MC1 for \( a/h \leq 0.83 \). Wave speed of MC2 extremely agrees with that of UPT for \( a/h \leq 0.6 \) by a deviation of about 0.001. However, wave speed of MC2 for \( a/h > 0.6 \) sharply increasingly differs from that of UPT.

From the above examination of mathematical validity of MC1 and MC2, MC2 shows better
satisfaction of Bernoulli’s equation and agreement with exact solution on wave profiles and wave speeds for $a/h \leq 0.83$ than MC1. Omitted high-order terms in a series are supposed too trivial to be neglected for an accurate approximation. If subsidiary constants $a_i$ ($i=3, 5, 7...$) are significantly smaller than the leading one $a_1$, the obtained approximation are almost approximate to the exact solution.

1.1 1.2 1.3 1.4
$c/c_0$

Fig. 5 Comparison of wave speed with that of UPT for $a/h=0.75$.

Figure 6 plots the ratio of $a_3$ to $a_1$ for all wave amplitudes. It is seen from Fig. 6 that the ratio gradually increases from small waves and reaches a maximum of 0.047 when $a/h=0.45$, and then rapidly drops down to zero when $a/h=0.795$. The result of small subsidiary ratio explains that MC2 can modifies MC1 with small corrections on wave profile and wave speed for $a/h \leq 0.6$ and McCowan’s form has fast convergent expression for solitary waves until $a/h=0.8$.

4. Horizontal drift

The total drift on a streamline $\varphi$ due to the passage of a solitary wave can be exactly defined as

$$\delta = \int_{x}^{x+\varphi} \frac{c + u(x, y(x, \varphi))}{u(x, y(x, \varphi))} dx$$  \hspace{1cm} (8)

The integral for wave’s drift depending on the horizontal velocity in steady state was proposed by Fenton (1972) who considered a solitary wave propagating in the negative x-axis. On the integration of Eq. (8), when $\varphi$ is specified a constant, the y-value should be first computed using Eq. (1) at any $x$ and then inserted into $u(x, y(x, \varphi))$. The adaptive integration with the trapezoidal rule for the improper integral is used for fast calculation in the paper. It is obvious from Fig. 1 that the surface elevation of a solitary wave near the crest varies faster that that far from the crest. Adaptive integration automatically allows for different panels on different subintervals, choosing values adequate for a specified accuracy by a absolute difference between two successive integrations less than $10^{-4}$.

The integral of Eq. (8) over time is improper due to the upper infinite limits. A large enough value of $x$ for both limits should be examined for the improper integration. An examination of computed drift against different $x$ intervals for three chosen waves shows that the integration approaches a convergence as its upper limit, $x/h$, is larger than 5.

A comparison of horizontal drifts of a particle at the surface computed by three approximations with that by Fenton’s approximation is plotted in Fig. 7. Horizontal drifts by MC1 and MC2 are larger than those by Fenton’s approximation. Horizontal drifts MC2 have smaller deviations from those of Fenton’s approximation for all wave amplitudes than MC1.

The bottom drift obtained by MC1, MC2 and Fenton’s approximations shows that the particle drifts at the bottom using MC1 and MC2 exceeds those of Fenton. Obtained particle drifts at mean depth using MC2 and Fenton’s ninth-order approximation for $a/h = 0.8$ are 1.7833 and 1.6748, respectively.
5. Conclusions

A second approximation to McCowan’s form for solitary waves is obtained in this paper using collocation method and compared with the first approximation of Chang and Liou (2013). The accuracy of proposed second order approximation was examined valid by comparing wave profiles and wave speeds of solitary waves with $a/h \leq 0.75$ with available high order solutions. Small subsidiary ratio of the second-order approximation implies that McCowan’s form has fast convergent expression for solitary waves until $a/h < 0.8$.

Good agreement of particle drifts at the surface, the mean depth and on the bottom compared with those of Fenton’s ninth-order approximations shows applicability of the second-order approximation.

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References