The fourth-order Boussinesq equations for internal waves in a two-fluid system

Chi-Min Liu¹

¹General Education Center, Chienkuo Technology University

ABSTRACT

The fourth-order Boussinesq equations for internal waves propagating in a two-fluid system are presented. Using the idea elucidated by Liu et al. (2008) a set of model equations is shown with the interfacial displacement and two velocity potentials at arbitrary elevations. For the sake of extending the applicable range from shallower water to deeper water, the Padé approximation was adopted to determine the optimal model. Viscous effects in both fluids will be considered to simulate the real ocean in the coming future.

Keywords: Boussinesq equations; two-fluid system; Padé approximation

1. Introduction

For a single-fluid system with free surface, the well-known classical Boussinesq equations were extended to the modified Boussinesq equations (MBE) which have been comprehensively studied for over two decades. MBE attracted a great deal of attention from all aspects of wave dynamic researchers due to academic values as well as wide practical applications specially in simulating waves propagation in coastal zones. Traditionally, classical Boussinesq equations (BE) adopted the depth-integrated velocity to be the variable associated with the surface elevation to form a set of equations. Due to the requirement of length scales of BE is \( a_0/h_0 = O(k_0 h_0^2) < 1 \) where \( a_0 \) is the typical amplitude, \( h_0 \) the typical water depth and \( k_0 \) the characteristic wave number, poor simulation usually occur while simulating waves in deeper-water regions. Thus many efforts were made to derive the MBE and to improve the capacity to overcome above scale limitations.

The most popular and well-known mathematical method is to overcome the length-scale limitations is the Padé approximation. Using this method and
modifying the depth-integrated velocity to a flexible velocity potential at specific elevation, MBE were be derived and shown in the following papers: Nwogu (1993), Chen and Liu (1995), Wei et al. (1995), Madsen and Schäffer (1998), Gobbi et al. (2000) and Madsen et al. (2002).

For internal waves in a two-fluid system, Liu et al. (2002) followed the idea of developing MBE for a single-fluid system and then derived a set of MBE for waves in a two-fluid system. Based on paper of Liu et al.(2008), a set of fourth-order Boussinesq equations for a two-fluid system bounded by two rigid boundaries is presented in Section 2. Using the Pade approximation the optimal model is determined and analyzed in Section 3. Results and comparison are presented in Section 4 and concluding remarks are made in Section 5.

2. Derivative of the fourth-order Boussinesq equations

In this paper we consider a two-fluid system bounded by two rigid, impermeable and horizontal plates. Both fluids are assumed to be inviscid and immiscible. Densities, velocity potentials and undisturbed thicknesses of both layers are denoted as \( \rho_1 \), \( \Phi_1 \) and \( h_1 \) with the subscripts 1 and 2 for upper and lower fluids, respectively. With the help of non-dimensional scales defined by Liu et al. (2008), the governing equations and boundary conditions are shown as below

\[
\begin{align*}
\mu^2 \nabla^2 \Phi_1 + \Phi_{1,zz} &= 0 \text{ within } \epsilon \eta \leq z \leq h_1, \\
\mu^2 \nabla^2 \Phi_2 + \Phi_{2,zz} &= 0 \text{ within } -h_1 \leq z \leq \epsilon \eta, \\
\Phi_{1,z} &= 0 \text{ at } z = h_1, \\
\Phi_{2,z} &= \mu \left( \eta + \epsilon \nabla \Phi_1 \cdot \nabla \eta \right) \text{ at } z = \epsilon \eta, \\
\Phi_{1,z} &= \mu \left( \eta + \epsilon \nabla \Phi_2 \cdot \nabla \eta \right) \text{ at } z = \epsilon \eta, \\
\Phi_{2,z} &= 0 \text{ at } z = -h_1, \\
\rho_1 \left[ \Phi_{1,z} + \frac{\epsilon}{2} \left( \nabla \Phi_1 \right)^2 + \frac{1}{\mu^2} \left( \Phi_{1,z} \right)^2 \right] &= \Phi_{2,z} + \frac{\epsilon}{2} \left( \nabla \Phi_2 \right)^2 + \frac{1}{\mu^2} \left( \Phi_{2,z} \right)^2 \text{ at } z = \epsilon \eta, \\
\end{align*}
\]

where the subscripts , , , and , represent the derivatives with respect to , and , respectively. The symbol \( \nabla \) is defined as \( \nabla = \left( \partial_z, \partial_x \right) \) and \( \rho_1 = \rho_f \rho_z \) the density ratio. Two parameters, \( \mu \) and \( \epsilon \), which measure the dispersive effect and the nonlinearity respectively, are defined as \( \mu = k_0 h_0 \) and \( \epsilon = a_0 h_0 \). Integrating Eqs.(1) and (2) with respect to \( z \) and taking boundary conditions into account, equations of mass conservation for both fluids are

\[
\begin{align*}
\nabla \cdot \left[ \int_{-h_1}^h \nabla \Phi_1 dz \right] &= \eta, \\
\nabla \cdot \left[ \int_{-h_1}^h \nabla \Phi_2 dz \right] &= -\eta.
\end{align*}
\]

By expanding both velocity potentials in terms of \( \mu \) and solving Eqs.(1) and (2) associated with boundary conditions Eq.(3) and (6), the following relations are obtained

\[
\begin{align*}
\Phi_1 &= \Phi_2 + \mu \left( \frac{z_1^2 - z_2^2}{2} + (z - z_t) h_1 \right) \nabla^2 \Phi_1, \\
&+ \mu^4 \left[ \frac{1}{24} z_1^4 - \frac{h_1}{6} z_1^3 + \left( \frac{z_1 h_1^2 - z_2^2}{4} \right) z_1^3 \right. \\
&\left. + \left( \frac{z_1^2 h_2^2}{2} - z_1^2 h_2 + \frac{h_1^2}{3} \right) z_1^2 + \frac{5}{24} z_1^4 - \frac{5}{6} z_1^2 h_2 + z_1^2 h_1^2 - \frac{1}{3} z_1^3 h_1^3 \right] \\
&\nabla^2 \nabla^2 \Phi_1 + O(\mu^6)
\end{align*}
\]

\[
\begin{align*}
\Phi_2 &= \Phi_1 + \mu \left( \frac{z_1^2 - z_2^2}{2} + (z - z_t) h_2 \right) \nabla^2 \Phi_2, \\
&+ \mu^4 \left[ \frac{1}{24} z_1^4 - \frac{h_2}{6} z_1^3 - \left( \frac{z_1 h_2^2 - z_2^2}{4} \right) z_1^3 \right. \\
&\left. - \left( \frac{z_1^2 h_1^2}{2} + z_1^2 h_1 + \frac{h_1^2}{3} \right) z_1^2 + \frac{5}{24} z_1^4 + \frac{5}{6} z_1^2 h_1 + z_1^2 h_2^2 + \frac{1}{3} z_1^3 h_2 \right] \\
&\nabla^2 \nabla^2 \Phi_2 + O(\mu^6)
\end{align*}
\]

where the subscripts , and , represent the elevations in the upper and lower layers on the specific elevations \( z = z_t \) and \( z = z_e \), respectively. Substituting Eqs.(10) and (11) into Eqs.(8) and (9) results in a set of Boussinesq equations.
from Eqs. (12) to (14), it gives two independent equations which are shown below.

\[
\begin{align*}
-\eta + h \frac{h^2}{5} + \mu \left( \frac{z h_t}{4} - z h_t + \frac{b_t}{3} \right) & = 0, \\
+ \mu \left( \frac{5}{24} z h_t - \frac{5}{6} z h_t^2 + \frac{7}{6} z h_t + \frac{2}{3} z h_t + \frac{2}{15} h_t \right) & = 0, \\
- \alpha^2 \cdot (\eta^2 \Phi_s^2) & = - \frac{2}{2} \left( \frac{z h_t}{2} - z h_t \right) \cdot \eta \cdot \frac{\Phi_s}{\eta} + O(\mu^3, \eta^2 \mu^3, \eta^3), \\
+ \mu \left( \frac{5}{24} z h_t - \frac{5}{6} z h_t^2 + \frac{7}{6} z h_t + \frac{2}{3} z h_t + \frac{2}{15} h_t \right) & = 0, \\
+ \mu \left( \frac{5}{24} z h_t - \frac{5}{6} z h_t^2 + \frac{7}{6} z h_t + \frac{2}{3} z h_t + \frac{2}{15} h_t \right) & = 0, \\
- \alpha^2 \cdot (\eta^2 \Phi_s^2) & = - \frac{2}{2} \left( \frac{z h_t}{2} - z h_t \right) \cdot \eta \cdot \frac{\Phi_s}{\eta} + O(\mu^3, \eta^2 \mu^3, \eta^3).
\end{align*}
\]

Equations (12) to (14) are the set of fourth-order Boussinesq equations for a rigid-lid case. If the upper boundary condition is changed to be a free surface, one can follow the same procedures to derive a set of four equations for a free-surface case.

3. Padé approximation and the optimial model

In this section, the method to extend the applicable range of MBE is the Padé approximation. Padé approximation is the excellent approximation of a function by a rational function of given order. A Padé approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge.

For MBE, the characteristic which the Padé approximation will be applied to is the linear dispersion relation. For linear waves, we assume both velocity potentials \( \Phi_s \) and \( \Phi_b \) are proportional to \( \exp[i(x - t)] \) and eliminate \( \eta \) from Eqs. (12) to (14). This gives the linear dispersion relation

\[
\omega^2 = \frac{(1 - \rho_s h_t) + \rho_s h_t}{h_t + \rho_s h_t} + \frac{1}{1 + \rho_s h_t} + \frac{1}{1 + \rho_s h_t} + \frac{1}{1 + \rho_s h_t},
\]

where \( \rho_s, \rho_s, \rho_s, \) and \( \rho_s \) are functions of \( h_t, h_t, z_s, z_s, \) and \( \rho_s, \). Since the derivation for model equations is based on the assumption \( O(\mu^3) \ll 1 \), all possible values of \( z_s \) and \( z_s \) will make the present equations behave excellent in shallow water configuration. If one hope to extend the applicable range to deeper water configuration, it requires to choose \( z_s \) and \( z_s \) carefully. The mathematical method used herein is to make the linear dispersion relation of the present model to be identical to that of the linear wave theory expressed in the form of Padé approximation. First, the linear dispersion relation of linear wave theory is (see Lamb 1932)

\[
\omega^2 = \frac{1}{\mu} \left( \frac{(1 - \rho_s) h_t + \rho_s h_t}{h_t + \rho_s h_t} \right) \tan(h_t, \mu) \tan(h_t, \mu) + \frac{1}{\mu} \tan(h_t, \mu) \tan(h_t, \mu),
\]

which can be rewritten in the form of the (4,4) Padé approximants

\[
\omega^2 = \frac{(1 - \rho) h_t + \rho h_t}{h_t + \rho h_t} + \frac{1 + \rho_1 h_t}{1 + \rho_1 h_t} + \frac{1 - \rho_1 h_t}{1 - \rho_1 h_t},
\]

where \( \rho_1, \rho_1, \rho_1,\) and \( \rho_1 \) are also functions of \( h_t, h_t, z_s, z_s, \) and \( \rho_s \). Now we match Eqs. (15) and (16), it gives two independent equations

\[
 \begin{cases}
 p_2 = C_2, \\
p_4 = C_4,
 \end{cases}
\]

which are shown below

\[
\begin{align*}
\frac{z^2}{2} - z h_t + \frac{b_t}{3} + h_t^2 + h_t^2 + h_t^2 - h_t^2 + h_t^2 & = 0, \\
+ \frac{h_t^2 + \rho h_t h_t^2 + h_t^2 + h_t^2}{9 h_t^2 + 6 \rho h_t h_t^2 + h_t^2 + h_t^2} + 9 \rho h_t^2 & = 0.
\end{align*}
\]
After parameters $z_a$ and $z_b$ are determined, the optimal mode is then obtained.

4. Results

Based on the optimal model equations with the optimal elevation parameters solved by Eqs. (19) and (20), linear dispersion relation and particle velocities can be theoretically calculated from the optimal model. Figure 1 shows the dispersion relations predicted by the fourth-order MBE (dash line), the second-order MBE (solid line) and the Padé solution (dash-dot line) for the case $h = h_i = 1$ and $\rho = 0.99$. For $\mu$ goes larger (deeper-water region), the fourth-order solution behaves better than the second-order model. It is quite straightforward because a higher-order Padé approximation gives a better result for larger regions. A higher Padé approximation results in a better result while $\mu$ goes large.

Next, particle velocities for cases of $\mu = 1$ and $\mu = 3$ are respectively shown in Figures 2 and 3. Bold-solid and bold-dash lines represent horizontal and vertical velocities of the fourth-order MBE while solid and dash lines denote the exact linear solutions.
It shows that, for smaller $\mu$ value, the behaviors of both horizontal and vertical velocities are much closer to the exact linear solutions. Moreover, the horizontal velocity always has a better performance than the vertical velocity for the same condition (see Liu et al. (2008) for details). In comparison with dispersion relation, particle velocities predicted by present model shows a poorer behavior. This can be overcome by introducing higher Padé approximation into the model equations.

5. Concluding remarks

In this paper, the $O(\mu^4)$ MBE for internal waves in a two-fluid system is derived. The optimal model is determined by choosing the free parameters with the help of Padé [4,4] approximation. Based on the derived equations, linear dispersion and particle velocities are preliminarily analyzed. The viscous effects will be included into MBE in the future.

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References