An asymptotic approach for solving Biot’s equations

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ABSTRACT

The bed form deformation has attracted significant interests from scientists and engineers for more than a hundred years for it is relative to many important engineering applications, such as the evolution of the shoreline morphology, the foundation stability of the underwater structures, the waterway safety and the dam dredging. Therefore, in the present study we aim at modeling the slowly deforming bed forms which is crucial to the slow bed form deformation. To model the slowly deforming bed forms, we employed the Biot’s equations (Biot (1956)) as the momentum equations of the bed. However, the Biot’s equations were built for investigating the characteristics of the propagation of acoustic waves in poroelastic medium and the characteristic scales of the acoustic waves are very different from the ones of the water waves/flow and the deformation of bed forms. Therefore, the equations shall be modified with the order of the magnitude analysis for excluding the negligible terms. As for the boundary conditions, we employed the simplified boundary conditions derived by Hsieh et al. (2001). After the suitable formulations are set, we analyze the time scales in the system which was observed from the computational results of Shih (1998). By utilizing the time scale differences (the order of the time scale of water is \( O(t_1) = O(L/U) \) and the order of the time scale of bed form deformation is \( O(t_2) = O(LG/p_0U^3) \)), we constructed a leading order analysis for analyzing the water/soil interaction.

Keywords: slowly deforming bed forms; asymptotic approach; Biot’s equations
1. Introduction

The bed form deformation has attracted significant interests from scientists and engineers for more than a hundred years for it is relative to many important engineering applications, such as the evolution of the shoreline morphology, the foundation stability of the underwater structures, the waterway safety and the dam dredging. Therefore, in the present study we aim at modeling the slowly deforming bed forms which is crucial to the slow bed form deformation. To model the slowly deforming bed forms, we employed Biot’s equations (see Biot 1956) as the momentum equations of the bed. The momentum equations include the elastic deformation term, the Darcian flow term and the excess pore water pressure. Yamamoto et al. 1978 solved the equations simultaneously for analyzing the perturbed pore pressure induced by linear water waves. Mei and Foda 1981 indicated that there is a boundary later exist at the boundary of the porous medium and thus introduced a perturbation analysis for the solutions. Huang and Chwang 1990 proposed an idea of solving the equations with Helmholtz decomposition and the idea have been employed by Song 1993, Chiang 1996, Shih 1998, and Hsieh 1999. Yet the solution process with Helmholtz decomposition is rather complicated. In this study, we utilized the time scale differences between water flow and bed form deformation and analytically solved the equations with an asymptotic approach. The formulations used in this study are introduced in section 2, the asymptotic approach is introduced in section 3 and a brief conclusion is made in section 4.

2. Formulations

The complete set of the momentum equations of the soil were derived by Biot (1956) for studying the propagation of acoustic waves in poroelastic materials. The acoustic waves are in general with high frequencies and short wave lengths while in this study we focus on the bed deformation which is a long term process. Therefore, some terms of the momentum equations are believed negligible for the present study. By assigning the order of time to be $T$, the perturbed pressure to be $P$, the length scale $L$ and the displacement of the water and the skeleton of the soil, according to the momentum equations, are $PT/bL$ and $PL/G$. For the case $(G/K) \ll 1$, the soil was defined as the “soft” and the soil deforms with the water flow or waves while a boundary layer liked area exists near the surface of the soil. The size of the soil particles are in general much smaller in “soft” material than the “hard” materials and the small particles (according to the Hazen’s equation (see Das, 1979)) lead to the small permeability and thus the Darcian flow inside the soil is negligible (see Song, 1993 and Hsieh, 1999). The shear modulus $G$ for silt is about $10^5$-$10^6$ N/m$^2$, the order of the coefficient of permeability $b$ is about $10^{11}$, the length scale in this study is about $10^0$-$10^2$ m and the time scale is about $10^0$-$10^3$ second. Therefore, it is reasonable to assume that the parameter $GT/bL^2$ is negligible in the present study. The continuity equation of the soil may be rewritten as, following with the assumptions of $(G/K) \ll 1$ and $(GT/bL^2) \ll 1$, the momentum equation is written as

$$\nabla \cdot \mathbf{d} = 0 \quad (1)$$

and the governing equation of the excess pore water pressure is found to be a Laplace’s equation and it is written as

$$\nabla^2 d = \nabla^2 p_2 = 0 \quad (3)$$

Equation (2) is the static equilibrium equation of a poroelastic medium which is similar to the equations derived by Biot (1941) for estimating the subsidence. The Laplace equation for modeling the excess pore
water pressure was originally adopted by Putnam (1949) and Reid and Kajiura (1957) (See Song, 1993) and the applicability of the Laplace equations is clarified in the present study. Hsieh et al. (2001) specified that the normal stress for soft material is only continuous inside the thin boundary layer which was proposed by Mei and Foda (1981) and is absent outside the layer, i.e. \( \tau_{xz} = 0 \) outside the boundary layer. Together with the simplified governing equations and the simplified boundary conditions of Hsieh et al. (2001), the analytical approach is conducted in the next section.

3. Asymptotic approach

To proceed the analysis, we employ the linearized formulations and introduce the potential function \( \phi_1 \) and the displacement potential function \( \psi_2 \) for simplicity. The relationships between the variables and the functions are given as

\[
\begin{align*}
\psi_1 & = \frac{\partial \phi_1}{\partial x} + u \\
\psi_2 & = \frac{\partial \phi_1}{\partial z} \\
\phi_2 & = \frac{\partial \phi_1}{\partial z} \\
\phi_3 & = -\frac{\partial \psi_2}{\partial x}
\end{align*}
\]

Consider the case of linear water waves passing over the porous medium, the exact orders of variables are obtained from the analytical solutions

\[
\begin{align*}
\text{O}(\phi_1) & \sim O\left(\frac{a}{k}\right) \\
\text{O}(\psi_2) & \sim O\left(\frac{a^2 \rho}{k^2 G}\right) \\
\text{O}(p_2) & \sim O\left(\frac{a^2 \rho}{k}\right) \\
\text{O}(\eta_2) & \sim O\left(\frac{a^2 \rho}{k^2 G}\right) \\
\text{O}(\xi) & \sim O\left(\frac{1}{\sqrt{gk}}\right) \\
\text{O}(\eta_1) & \sim O(\alpha)
\end{align*}
\]

The order of variables for the case of linear water waves passing over a soil bed presented in equations (8) to (14) implied the applicability of a leading order analysis for the variables of the soil \( (\psi_2 \text{ and } \eta_2) \) are of the order \( O(1/G) \) while the shear modulus \( G \) of silt is about \( 10^9 \text{ N/m}^2 \) and thus the variables of soil are much smaller than the ones of water. By introducing the order of variables into the formulations, the K.F.S.B.C of the soil can be written as

\[
\left(\frac{\partial \eta_1}{\partial x}\right) + \left(\frac{\partial \eta_2}{\partial z}\right) = \frac{\partial \phi_1}{\partial z}
\]

(15)

in which the notation " \( \sim \) " stands for non-dimensional variables. For the case \( O(gp/kG) \ll 1 \), the K.F.S.B.C turn into

\[
\frac{\partial \phi_1}{\partial x} = 0
\]

(16)

The water part of the problem with (16) is reduced to a linear wave problem with impermeable bed and the formulations of the water part are written as follows.

Continuity equations of water at the domain

\( 0 < \tilde{z} < kh_1 \), \( -\infty < \tilde{x} < \infty \) is

\[
\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0
\]

(17)

The linearized K.F.S.B.C. and D.F.S.B.C. of water at \( \tilde{z} = kh_1 \) are

\[
\frac{\partial \tilde{\eta}_1}{\partial t} + \frac{1}{\rho} \frac{\partial \tilde{\eta}_1}{\partial x} = \frac{\partial \tilde{\phi}_1}{\partial z}
\]

(18)

\[
\frac{\partial \tilde{\phi}_1}{\partial t} + \frac{1}{\rho} \frac{\partial \tilde{\phi}_1}{\partial x} + \tilde{\eta}_1 = 0
\]

(19)

The linearized K.F.S.B.C. soil at \( \tilde{z} = 0 \) is

\[
\frac{\partial \tilde{\phi}_1}{\partial x} = 0
\]

(20)

The linkage between water and soil relied on the continuity of pressure. The continuity of pressure at \( \tilde{z} = 0 \) is

\[
\tilde{p}_2 = \tilde{p}_1
\]

(21)

and the computations of water and soil are thus decoupled. The formulations of soil are written as follows:

Continuity equation of soil at the domain \( -\infty < \tilde{z} < 0 \), \( -\infty < \tilde{x} < \infty \) is
\[ \frac{\partial^4 \psi_2}{\partial z^4} + 2 \frac{\partial^4 \psi_2}{\partial x^2 \partial z^2} + \frac{\partial^4 \psi_2}{\partial x^4} = 0 \]  
(22)

Governing equation of pressure at the domain 
\(-\infty < \hat{z} < 0 \), \(-\infty < \hat{x} < \infty \) is
\[ \frac{\partial^2 \bar{p}_2}{\partial z^2} + \frac{\partial^2 \bar{p}_2}{\partial x^2} = 0 \]  
(23)

Momentum equations of soil at the domain \(-\infty < \hat{z} < 0 \), \(-\infty < \hat{x} < \infty \) are
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial \psi_2}{\partial z} = \frac{\partial \bar{p}_2}{\partial z} \]  
(24)
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \left( - \frac{\partial \psi_2}{\partial z} \right) = \frac{\partial \bar{p}_2}{\partial x} \]  
(25)

The linearized K.F.S.B.C. and D.F.S.B.C. of water at \( \hat{z} = h_1 \) are
\[ \frac{\partial \psi_1}{\partial t} + 0 \frac{\partial \bar{\eta}_1}{\partial z} = \frac{\partial \phi_1}{\partial \hat{z}} \]  
(26)
\[ \frac{\partial \psi_1}{\partial t} + 0 \frac{\partial \bar{\eta}_1}{\partial \hat{z}} + \bar{\eta}_1 = 0 \]  
(27)

The linearized K.F.S.B.C. soil at \( \hat{z} = 0 \) is
\[ \bar{\eta}_2 = - \frac{\partial \bar{p}_2}{\partial \hat{z}} \]

The free shear stress at \( \hat{z} = 0 \) is
\[ \frac{\partial^2 \bar{\psi}_2}{\partial \hat{z}^2} = 0 \]  
(28)

The variables vanish at far field, i.e. at \( \hat{z} = -\infty \), are
\[ \bar{\psi}_2 \rightarrow 0 \]  
(29)
\[ \bar{p}_2 \rightarrow 0 \]  
(30)

The analytical solutions of the leading order formulations are derived in Appendix A and the solutions are written as
\[ \phi_1 = a_1 [\coth\hat{h}_1 \cosh(\hat{z} - \hat{h}_1) + \sinh(\hat{z} - \hat{h}_1)] \exp[i(kx - \omega t)] \]  
(31)
\[ \psi_2 = -ia_1 \rho_1(\omega - Uk)^2 \left[ \frac{1}{k} \exp(\hat{z}k) + \exp(-\hat{z}k) \right] \]  
(32)
\[ \rho_2 = a_\rho \rho_1(\omega - Uk)^2 \left[ \frac{1}{k} \exp(\hat{z}k) + \exp(-\hat{z}k) \right] \]  
(33)
\[ \eta_2 = \frac{\rho_1(\omega - Uk)^2}{[(\rho_2 + \rho_1)g - 2k^2G] \sinh h_1} \exp[i(kx - \omega t)] \]  
(34)

Comparing the solutions to Song (1993), the solutions obtained from the leading order formulations are much simpler. A verification of the leading order solutions are presented as follows.

Table 1 The parameters of water and soil employed for verifying the leading order algorithm

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Quantity</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>amplitude ( a )</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>water depth ( h_1 )</td>
<td>10</td>
<td>m</td>
</tr>
<tr>
<td>wave number ( k )</td>
<td>0.2</td>
<td>1/m</td>
</tr>
<tr>
<td>water density ( \rho_w )</td>
<td>1000</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>porosity ( n_0 )</td>
<td>0.445</td>
<td></td>
</tr>
<tr>
<td>soil density ( \rho_s )</td>
<td>2692</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>shear modulus ( G )</td>
<td>1x10(^6)</td>
<td>N/m(^2)</td>
</tr>
<tr>
<td>elastic modulus of water ( K )</td>
<td>10(^9)</td>
<td>N/m(^2)</td>
</tr>
</tbody>
</table>

Figure 1 Verification of the potential of water
4. Conclusion

In this study, we obtained the suitable formulations for modeling the slowly deforming bed by simplifying the governing equations via the order of magnitude analysis and by employing the simplified boundary conditions derived by Hsieh et al. (2001) with the assumption of soft material. By utilizing the order difference between the variables of water and soil, we constructed a leading order analysis and we verified the a leading order analysis with the exact solutions of harmonic waves and obtained satisfactory results.

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